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On convex sublattices of distributive lattices

by

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1. Introduction

In this paper we study some properties of convex sublattices of distributive lattices.

The family of all convex sublattices of a lattice L will be denoted by $\mathcal{C}(L)$.

Section 2 contains some definitions and preliminary lemma's.

The first result of section 3 is the following: Let L be a distributive lattice and let $A, B \in \mathcal{C}(L)$ with $A \subset B$, $A \neq \emptyset$. In theorem 1 we prove that the family of all elements of $\mathcal{C}(L)$ which have the intersection A with B has a largest element, by means of an explicit construction of this element from A and B. The next theorems are concerned with congruence relations. Let $C \in \mathcal{C}(L)$. We construct the smallest congruence relation such that C is one of its congruence classes, and the largest congruence relation such that all elements of C are incongruent with respect to this congruence relation. Next, these results are related to the construction of theorem 1.

In section 4 we consider the lattice $(\mathcal{C}(L), \subset)$, i.e. the family of all convex sublattices of L, partially ordered by inclusion. We prove that in a distributive relatively complemented lattice L, all intervals $[\emptyset, A]$ of $(\mathcal{C}(L), \subset)$ are complemented. A necessary and sufficient condition that $(\mathcal{C}(L), \subset)$ be relatively complemented, is that L is also discrete (i.e., all intervals of L have finite lenght).

In section 5 we introduce an ordering $\leq$ on $\overline{\mathcal{C}}(L)$ (i.e., the family of all non-empty convex sublattices of L), which is a variant of the ordering by inclusion. We prove that $(\overline{\mathcal{C}}(L), \leq)$ is a distributive lattice, if L is distributive. Next we consider the lattices $\overline{\mathcal{C}}^2(L) = \overline{\mathcal{C}}(\overline{\mathcal{C}}(L))$, ... , $\overline{\mathcal{C}}^i(L)$. We prove that $\overline{\mathcal{C}}^i(B_j)$, where B_j is the Boolean algebra with 2^j elements, is isomorphic with the direct union of j factors F_i, where F_i is the free distributive lattice with i generators, with an extra zero and unit element adjoined.

Section 6 is concerned with a ternary function which can be used to characterize convex sublattices of distributive relatively complemented lattices. Finally, we exhibit a set of axioms for distributive relatively complemented lattices in terms of this ternary function.

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2. Definitions

Definition 1. Let X be a subset of a lattice L. The sets X_l, X_r are defined as follows:

$$X_l = \{ a \in L \mid \exists x \in X \text{ such that } a \leq x \},$$
$$X_r = \{ a \in L \mid \exists x \in X \text{ such that } a \geq x \}.$$

It is easily seen that:

1. l and r are closure-operators, i.e. for all X, $Y \subset L$ we have: $X \subset X_l$, $X_l = X_{l.l}$, $(X \cup Y)_l = X_l \cup Y_l$, and similarly for r.
2. $X_{lr} = X_{rl} = L$.
3. If X is closed with respect to $\vee (\wedge)$ then $X_l (X_r)$ is a $\vee$-ideal ($\wedge$-ideal).
4. If X is a $\vee$-ideal ($\wedge$-ideal) then $X = X_l$ $(X = X_r)$.

Definition 2. Let X, Y be non-empty subsets of a lattice L. The sets $X \wedge Y$, $X \vee Y$ are defined as follows:

$$X \wedge Y = \{ x \wedge y \mid x \in X \text{ and } y \in Y \},$$
$$X \vee Y = \{ x \vee y \mid x \in X \text{ and } y \in Y \}.$$

Clearly, for all X, $Y \subset L$ we have $(X \wedge Y)_l = X_l \wedge Y_l$, and $(X \vee Y)_r = X_r \vee Y_r$. It is also easy to prove that $(X \vee Y)_l = X_l \vee Y_l$, for all X, $Y \subset L$, if and only if L is distributive (and dually).

Definition 3. A subset of a lattice L is called convex if and only if $X_l \cap X_r \subset X$.

In this paper we are only interested in convex sublattices. The family of all convex sublattices of L will be denoted by $\mathcal{C}(L)$. The family of all $\vee$-ideals ($\wedge$-ideals) of L will be denoted by I(L) (Y(L)). Some of the simplest properties of $\mathcal{C}(L)$ are:

1. $I(L) \subset \mathcal{C}(L)$ and $J(L) \subset \mathcal{C}(L)$.
2. The intersection of a family of convex sublattices is a convex sublattice.
3. If A is closed with respect to $\vee$ and B is closed with respect to $\wedge$ then $A_l \cap B_r \in \mathcal{C}(L)$.
4. A subset C of L is a convex sublattice of L if and only if it has the following property: For all c_1, $c_2 \in C$ and all $x \in L$ we have: $c_1 \wedge (x \vee c_2) \in C$ and $c_1 \vee (x \wedge c_2) \in C$.

Clearly, if $C \in \mathcal{C}(L)$ then $C = C_l \cap C_r$. Hence, each convex sublattice can be written as the intersection of a $\vee$-ideal and a $\wedge$-ideal. The following lemma proves that this "decomposition" is unique:

<u>Lemma 1.</u> Let $C \in \mathcal{C}(L)$, $C \neq \emptyset$, and suppose that $C = I \cap J$, where I is a $\vee$-ideal and J is a $\wedge$-ideal. Then $I = C_l$ and $J = C_r$.

<u>Proof</u>. $C = I \cap J \subset I$, hence $C_l \subset I_l = I$. Also, $I \vee (I \cap J) = C$; hence, $I = I_l \subset \{I \vee (I \cap J)\}_l = C_l$. Thus $I = C_l$. Similarly, $J = C_r$.

From this lemma it follows that if C, $D \in \mathcal{C}(L)$, and $C \cap D \neq \emptyset$, then $(C \cap D)_l = C_l \cap D_l$, and $(C \cap D)_r = C_r \cap D_r$.

<u>Definition 4.</u> Let C, $D \in \mathcal{C}(L)$. The smallest convex sublattice of L that contains C and D is denoted by $C \sqcup D$.

From this definition it follows that if $C \neq \emptyset$, $D \neq \emptyset$, than $C \sqcup D = (C \wedge D)_r \cap (C \vee D)_l$.

The following two lemma's state some properties of convex sublattices that will be used later.

<u>Lemma 2</u>. Let A, B, $C \in \mathcal{C}(L)$, with $A \cap B \neq \emptyset$, $B \cap C \neq \emptyset$, and $C \cap A \neq \emptyset$. Then $A \cap B \cap C \neq \emptyset$.

<u>Proof</u>. Let $x \in A \cap B$, $y \in B \cap C$ and $z \in C \cap A$. Since x, $y \in B$ and $x, z \in A$, we have $x \wedge (y \vee z) \in A \cap B$.

Now consider the element $\{x \wedge (y \vee z)\} \vee (y \wedge z)$.

We have: $x \wedge (y \vee z)$ and y are elements of B,

$x \wedge (y \vee z)$ and z are elements of A,

$y \vee z$ and $y \wedge z$ are elements of C.

Therefore, $\{x \wedge (y \vee z)\} \vee (y \wedge z) \in A \cap B \cap C$.

Two consequences of this lemma are:

1. If $A_i \in \mathcal{C}(L)$, $1 \leq i \leq n$, and $\bigcap_{i=1}^{n} A_i = \emptyset$, then $A_i \cap A_j = \emptyset$, for some i, j with $1 \leq i, j \leq n$.
2. If $A_i \in \mathcal{C}(L)$, $1 \leq i \leq n$, and $\bigcap_{\substack{i=1 \\ i \neq j}}^{n} A_i \neq \emptyset$ for three values of j $(1 \leq j \leq n)$, then $\bigcap_{i=1}^{n} A_i \neq \emptyset$.

Lemma 3. Let L be a distributive lattice and let C, $D \in \mathcal{C}(L)$ with $C \cap D = \emptyset$. Then there exist C', $D' \in \mathcal{C}(L)$, such that $C \subset C'$, $D \subset D'$, $C' \cap D' = \emptyset$ and $C' \cup D' = L$. Moreover, either C' is a $\vee$-ideal and D' is a $\wedge$-ideal or conversely.

Proof. (This proof is due to P.C. Baayen).

1. Either $C_l \cap D_r = \emptyset$, or $C_r \cap D_l = \emptyset$. For, suppose that there exist c_1, $c_2 \in C$ and d_1, $d_2 \in D$ with $c_1 \leq d_1$ and $c_2 \geq d_2$. Then $c_1 \leq d_1 \wedge (c_1 \vee d_2) \leq c_1 \vee c_2$; hence, $d_1 \wedge (c_1 \vee d_2) \in C \cap D$, a contradiction.
2. Suppose $C_l \cap D_r = \emptyset$. We can then apply Stone's theorem [10] to the $\wedge$-ideal D_r.

3. Congruence relations

Theorem 1. Let L be a distributive lattice, let A, $B \in \mathcal{C}(L)$ with $A \subset B$, $A \neq \emptyset$. Let C be defined as:

$C = (A_r \setminus (B \setminus A_l)_r)_l \cap (A_l \setminus (B \setminus A_r)_l)_r$.

Then:

1. $C \in \mathcal{C}(L)$.
2. $B \cap C = A$.
3. $D \in \mathcal{C}(L)$ and $D \cap B = A$ imply $D \subset C$.

Proof.

1. In order to prove that $C\in\mathcal{C}(L)$ it is sufficient to prove that $A_r\setminus(B\setminus A_1)_r$ is closed with respect to $\vee$. Let $a_1', a_2'\in A_r\setminus(B\setminus A_1)_r$. Clearly, $a_1'\vee a_2'\in A_r$. Suppose that $a_1'\vee a_2'\geq b$ for some $b\in B\setminus A_1$. Since $a_1'\in A_r$, there exists $a_1\in A$ such that $a_1'\geq a_1$.
   Then $b\geq b\wedge a_1'\geq b\wedge a_1$. Sine $A\subset B$ and B is a convex sublattice, we have $b\wedge a_1\in B$, and $b\wedge a_1'\in B$. Since $a_1'\geq b\wedge a_1$, and $a_1'\notin(B\setminus A_1)_r$, we see that $b\wedge a_1'\in A_1$, so that there exists $a_3\in A$, with $a_3\geq b\wedge a_1'$. Similarly, there exists $a_4\in A$ such that $a_4\geq b\wedge a_2'$. Thus, $a_3\vee a_4\geq(b\wedge a_1')\vee(b\wedge a_2')=b\wedge(a_1'\vee a_2')=b$, which contradicts $b\in B\setminus A_1$.
   We conclude therefore that $a_1'\vee a_2'\in A_r\setminus(B\setminus A_1)_r$, where $A_r\setminus(B\setminus A_1)_r$ is closed with respect to $\vee$ (it is easy to prove that $A_r\setminus(B\setminus A_1)_r$ is even a convex sublattice).

2.1. In order to prove that $A\subset B\cap C$ it is sufficient to prove that $A\subset A_r\setminus(B\setminus A_1)_r$. It is clear that $A\subset A_r$. Also, it is impossible that there exists $a\in A$ such that $a\geq b$ for some $b\in B\setminus A_1$.

2.2. Let $b\in B\cap C$. Then there exist $a'\in A_r\setminus(B\setminus A_1)_r$ and $a''\in A_1\setminus(B\setminus A_r)_1$ such that $a''\leq b\leq a'$. From $a'\geq b$ and $a'\notin(B\setminus A_1)_r$ we see that $b\in A_1$. Similarly, from $a''\leq b$ we infer that $b\in A_r$. Hence $b\in A_1\cap A_r=A$, from which we conlcude that $B\cap C\subset A$.

3. Suppose $D\in\mathcal{C}(L)$ and $D\cap B=A$. We have to prove that $D\subset C$. It is sufficient to show that for each $d\in D$ and $a\in A$: $d\vee a\in A_r\setminus(B\setminus A_1)_r$. Clearly, $d\vee a\in A_r$. Suppose that $d\vee a\geq b$ for some $b\in B\setminus A_1$. Then $b=b\wedge(d\vee a)\leq(b\wedge d)\vee a$. Since $A\subset B$ and $A\subset D$, we have $(b\wedge d)\vee a\in B$ and $(b\wedge d)\vee a\in D$; hence, $(b\wedge d)\vee a\in A$. This contradicts $b\in B\setminus A_1$.

Corollary. Let L be a distributive lattice, let $A, B\in\mathcal{C}(L)$ with $A\subset B$, $A\neq\emptyset$, and let $C(A, B)$ be the largest element of $\mathcal{C}(L)$ which has the intersection A with B. Then $C(A, B)=C(A, C(A, C(A, B)))$.

Proof. Since $C(A, C(A, B))$ is the largest convex sublattice which has the intersection A with $C(A, B)$, and since $B\cap C(A, B)=A$, we have $B\subset C(A, C(A, B))$. Thus, $B\cap C(A, C(A, C(A, B)))=B\cap C(A, C(A, B))\cap C(A, C(A, C(A, B)))=B\cap A=A$.
Since $C(A, B)$ is the largest convex sublattice which has the intersection A with B, we have

(1) $C(A, C(A, C(A, B))) \subset C(A, B)$.
Since $C(A, B) \cap C(A, C(A, B)) = A$, and since $C(A, C(A, C(A, B)))$ is the largest convex sublattice which has the intersection A with $C(A, C(A, B))$, we have

(2) $C(A, B) \subset C(A, C(A, C(A, B)))$.
From (1) and (2) the assertion follows.

Remarks:

1. From this corollary it follows that $C(A, C(A, B))$ is the largest element of the family of all elements $B' \in \mathcal{C}(L)$ such that $C(A, B) = C(A, B')$:
   a. If $B' = C(A, C(A, B))$ then $C(A, B') = C(A, C(A, C(A, B))) = C(A, B)$.
   b. If $C(A, B) = C(A, B')$, then $B' \subset C(A, C(A, B')) = C(A, C(A, B))$.
2. In section 4 we shall derive a sufficient condition for L in order that for each $A, B \in \mathcal{C}(L)$ with $A \subset B$, $A \neq \emptyset$, we have $B = C(A, C(A, B))$.
3. Clearly, the corollary can be formulated more generally as a statement on sets instead of on lattices.

The next theorems are concerned with congruence relations.
In theorems 2 and 3 we investigate some general properties of congruence relations in distributive lattices, and in theorem 5 we relate these properties to the construction of theorem 1.

<u>Theorem 2</u>. Let L be a distributive lattice and let K be a sublattice of L. Let the relation R_K be defined as follows:

xR_Ky if and only if there exist $k_1, k_2 \in K$ such that $k_1 \wedge x = k_1 \wedge y$ and $k_2 \vee x = k_2 \vee y$.

Then R_K is the smallest congruence relation that contains K in one of its congruence classes.

<u>Proof</u>. It is clear that xR_Kx and that xR_Ky implies yR_Kx. Now suppose that xR_Ky and yR_Kz hold. This means that there exist k_1, k_2, k_3, k_4 such that $k_1 \wedge x = k_1 \wedge y$, $k_2 \vee x = k_2 \vee y$, $k_3 \wedge y = k_3 \wedge z$ and $k_4 \vee y = k_4 \vee z$. Hence, $k_1 \wedge k_3 \wedge x = k_1 \wedge k_3 \wedge z$ and $k_2 \vee k_4 \vee x = k_2 \vee k_4 \vee z$. Since K is a sublattice, we see that xR_Kz. It is easy to verify that if $x\,R_Ky$ and

$t \in L$ than $x \wedge t R_K y \wedge t$ and $x \vee t R_K y \vee t$. Clearly, all elements of K are congruent with respect to R_K. There remains the proof that R_K is the smallest congruence relation with this property. Suppose S is a congruence relation such that for all $k_1, k_2 \in K$: $k_1 S k_2$.
We prove that $R_K \leq S$, i.e., $x R_K y$ implies $x S y$. From $x R_K y$ we see that there exist k_1, k_2 such that $k_1 \wedge x = k_1 \wedge y$ and $k_2 \vee x = k_2 \vee y$. From $k_1 S k_2$ it follows that $x \wedge k_1 \; S x \wedge k_2$; hence,
$y = y \vee (y \wedge k_1) = y \vee (x \wedge k_1) S \; y \vee (x \wedge k_2)$. also, $y \wedge k_1 \; S \; y \wedge k_2$; hence,
$x = x \vee (x \wedge k_1) = x \vee (y \wedge k_1) S \; x \vee (y \wedge k_2)$. Thus,
$y \; S \; y \vee (x \wedge k_2) = (y \vee x) \wedge (y \vee k_2) = (y \vee x) \wedge (x \vee k_2) = x \vee (y \wedge k_2) S \; x$.

Corollary. 1. Let a,b be two elements of a distributive lattice L, with $a \leq b$. The smallest congruence relation $R_{[a,b]}$ with the property that $[a,b]$ is one of its congruence classes, can be defined as follows:

$$x R_{[a,b]} y \text{ if and only if } a \wedge x = a \wedge y \text{ and } b \vee x = b \vee y.$$

2. Let I be a $\vee$-ideal of the distributive lattice L. The smallest congruence relation R_I which has I as one of its congruence classes can be defined as follows:

$$x R_I y \text{ if and only if there exists } i \in I \text{ such that } x \vee i = y \vee i.$$

Proof.
1. $a \wedge x = a \wedge y$ and $b \vee x = b \vee y$ is equivalent to the existence of two elements c_1, $c_2 \in [a,b]$ with $c_1 \wedge x = c_1 \wedge y$ and $c_2 \vee x = c_2 \vee y$. It can be verified directly that $[a,b]$ is a congruence class of $R_{[a,b]}$.
2. It is only necessary to prove that there exists $i_1 \in I$ with $i_1 \wedge x = i_1 \wedge y$. However, for each $i \in I$ we have $(i \wedge x \wedge y) \wedge x = (i \wedge x \wedge y) \wedge y$, and $i \wedge x \wedge y \in I$.

Remark: Grätzer and Schmidt [4] have given another definition of $R_{[a,b]}$ which requires a more complicated proof. Corollary 2 also occurs in [4], again with a more elaborate proof.

Theorem 3. Let L be a distributive lattice and let K be a sublattice of L. We define the relation θ_K as follows:

$$x \, \theta_K y \text{ if and only if for all } k_1, k_2 \in K : k_1 \wedge (x \vee k_2) = k_1 \wedge (y \vee k_2).$$

Then θ_K is a congruence relation such that different elements of K belong to different congruence classes of θ_K. If K is also convex, then θ_K is the largest congruence relation with this property.

Proof. It can be verified directly that θ_K is a congruence relation. Suppose $k_1 \ \theta_K \ k_2$ for some $k_1,k_2 \in K$. Then by the definition of θ_K:

$$k_1 \wedge (k_1 \vee k_2) = k_1 \wedge (k_2 \vee k_2) \text{ and}$$

$$k_2 \wedge (k_1 \vee k_1) = k_2 \wedge (k_2 \vee k_1).$$

Hence, $k_1 \leq k_2$ and $k_2 \leq k_1$. Thus, $k_1 = k_2$.
Suppose that K is also convex, and let θ^* be a congruence relation such that all elements of K belong to different congruence classes of θ^*. We prove that $\theta^* \leq \theta_K$. Let $x \ \theta^* \ y$. Then for all $k_1,k_2 \in K$:
$k_1 \wedge (x \vee k_2) \ \theta^* k_1 \wedge (y \vee k_2)$.
Since $k_1 \wedge (x \vee k_2) \in K$ and $k_1 \wedge (y \vee k_2) \in K$ we have $k_1 \wedge (x \vee k_2) =$ $k_1 \wedge (y \vee k_2)$, by the definition of θ^*.
This means that $x \theta_K y$.

Definition 5. Let L be a lattice. The zero element of the lattice of all congruence relations of L will be denoted by Ω, the unit element of this lattice will be denoted by $\mho$.

Corollary. Let K be a sublattice of a distributive lattice. Let R_K and θ_K be defined as in theorems 2 and 3. Then $R_K \wedge \theta_K = \Omega$.

Proof. Suppose $x \ R_K \wedge \theta_K y$, i.e., $x \ R_K y$ and $x \theta_K y$ both hold.
From $x \ R_K y$ it follows that there exist k_1,k_2 K such that $k_1 \wedge x = k_1 \wedge y$ and $k_2 \vee x = k_2 \vee y$. However, from $x \theta_K y$ we see that $k_2 \wedge (x \vee k_1) = k_2 \wedge$ $(y \vee k_1)$. Also, $k_2 \vee (x \vee k_1) = k_2 \vee (y \vee k_1)$. Since L is distributive, we have $x \vee k_1 = y \vee k_1$. Together with $x \wedge k_1 = y \wedge k_1$, this yields $x = y$.

For the proof of theorem 5 we need a theorem of J. Hashimoto.

Definition 6. A lattice is called discrete if and only if all its intervals have finite lenght.

<u>Theorem 4</u>. The lattice of all congruence relations of a lattice L is a Boolean algebra if and only if L is distributive and discrete.

<u>Proof</u>. See [6], theorem 8.4.

<u>Theorem 5.</u> Let L be a distributive lattice and let $C \in \mathcal{C}(L)$, $C \neq \emptyset$. For $c \in C$, let C_c be the largest convex sublattice of L which has the intersection $\{c\}$ with C. Let the relation Γ_C be defined as follows:

$$x \Gamma_C y \text{ if and only if there exists } c \in C \text{ such that } x \in C_c \text{ and } y \in C_c.$$

Then:

1. $C_{c_1} \cap C_{c_2} = \emptyset$, if $c_1 \neq c_2$.
2. If $x \in C_{c_1}$ and $y \in C_{c_2}$, then $x \wedge y \in C_{c_1 \wedge c_2}$ and $x \vee y \in C_{c_1 \vee c_2}$.
3. If C is an interval then Γ_C is a congruence relation.
4. If Γ_C is a congruence relation, then Γ_C is equal to the congruence relation θ_C as introduced in theorem 3.
5. If L is also relatively complemented then the following two assertions are equivalent:
   a) L is discrete.
   b) Γ_C is a congruence relation for each $C \in \mathcal{C}(L)$.

<u>Proof.</u>

1. Since $C \cap C_{c_1} \cap C_{c_2} = \{c_1\} \cap \{c_2\} = \emptyset$, and since $C \cap C_{c_1} = \{c_1\}$, $C \cap C_{c_2} = \{c_2\}$, we conclude that $C_{c_1} \cap C_{c_2} = \emptyset$, by lemma 2.
2. Let $x \in C_{c_1}$, $y \in C_{c_2}$. We only prove that $x \wedge y \in C_{c_1 \wedge c_2}$. By theorem 1, there exist $s \in \{c_1\}_r \setminus (C \setminus \{c_1\}_1)_r$, and $t \in \{c_2\}_r \setminus (C \setminus \{c_2\}_1)_r$, such that $x \leq s$ and $y \leq t$. We show that $s \wedge t \in \{c_1 \wedge c_2\}_r \; (C \setminus \{c_1 \wedge c_2\}_1)_r$. Since $s \geq c_1$ and $t \geq c_2$, we have $s \wedge t \geq c_1 \wedge c_2$. Suppose $s \wedge t \in (C \setminus \{c_1 \wedge c_2\}_1)_r$. This means that there exists $\bar{c} \in C$ such that $s \wedge t \geq \bar{c}$, but $\bar{c} \notin \{c_1 \wedge c_2\}_1$. As in the proof of theorem 1, we have: $c_1 \geq \bar{c} \wedge s$ and $c_2 \geq \bar{c} \wedge t$; hence, $c_1 \wedge c_2 \geq \bar{c} \wedge s \wedge t = \bar{c}$, a contradiction. Thus, $s \wedge t \in \{c_1 \wedge c_2\}_r \setminus (C \setminus \{c_1 \wedge c_2\}_1)_r$. Since $x \leq s$ and $y \leq t$ we have $x \wedge y \leq s \wedge t$, whence $x \wedge y \in (\{c_1 \wedge c_2\}_r \setminus (C \setminus \{c_1 \wedge c_2\}_1)_r)_1$. Similarly, it can be shown that $x \wedge y \in (\{c_1 \wedge c_2\}_1 \setminus (C \setminus \{c_1 \wedge c_2\}_r)_1)_r$. We conclude that $x \wedge y \in C_{c_1 \wedge c_2}$.

3. Let C be an interval, say $C = \{x \in L \mid a \le x \le b\}$. By 1 and 2, in order to prove that Γ_C is a congruence relation, we only have to show that $\bigcup_{c \in C} C_c = L$. By the maximality of the sets C_c, it is sufficient to show that for each $z \in L$ there exists a convex sublattice containing z, the intersection of which with C contains precisely one element. Let $D = \{y \in L \mid b \wedge z \le y \le a \vee z\}$. Then D has the required property: if $t \in C \wedge D$, then $a \le t \le b$ and $b \wedge z \le t \le a \vee z$; hence, $a \vee (b \wedge z) \le t \le b \wedge (a \vee z)$. Since L is distributive, we have $a \vee (b \wedge z) = t = b \wedge (a \vee z)$.
4. Let Γ_C be a congruence relation. Clearly, all elements of C belong to different congruence classes of Γ_C. By theorem 3, $\Gamma_C \le \theta_C$. We prove that also $\theta_C \le \Gamma_C$. Suppose $x\,\theta_C y$, and let $x \in C_{c_1}$, $y \in C_{c_2}$.

   Since $c_1 \wedge (x \vee c_2) = c_1 \wedge (y \vee c_2)$, we have $C_{c_1 \wedge (c_1 \vee c_2)} = C_{c_1 \wedge (c_2 \vee c_2)}$, i.e., $c_1 \wedge (c_1 \vee c_2) = c_1 \wedge (c_2 \vee c_2)$ or $c_1 \le c_2$.
   Similarly, $c_2 \le c_1$, from which $x\,\Gamma_C y$ follows.
5. Let L be distributive and relatively complemented.
   a. Suppose L is discrete. We prove that Γ_C is a congruence relation for each $C \in \mathcal{C}(L)$. As in 3, it is sufficient to show that for each $x \in L$ there exists a convex sublattice C^*, containing x, which meets C in precisely one point. Let $x \in L$. Consider the congruence relation R_C. By theorem 4, R_C has a complement R_C^*. Let c be an arbitrary element of C. Then $x R_C \vee R_C^* c$. Since L is relatively complemented, we have $x\, R_C^* R_C c$, i.e., there exists $t \in L$ with $x\, R_C^* t$ and $t\, R_C c$. Let C^* be the congruence class of R_C^* which contains both x and t. It follows that $C \cap C^* = \{t\}$; hence, C^* has the desired property. (We see that $R_C^* = \Gamma_C$; this can be shown as follows:
   α. By the corollary of theorem 3, we have $R_C \wedge \Gamma_C = \Omega = R_C \wedge R_C^*$.
   β. By 4, $R_C^* \le \Gamma_C$. Since $R_C \vee R_C^* = \mho$, we have $R_C \vee \Gamma_C = \mho = R_C \vee R_C^*$.
   γ. Since the lattice of all congruence relations of a lattice is distributive, we conclude that $\Gamma_C = R_C^*$).

   b. Suppose that L is distributive and relatively complemented and that Γ_C is a congruence relation for each $C \in \mathcal{C}(L)$. We prove that L is discrete. By theorem 4, it is sufficient to prove that each congruence relation of L has a complement. Let R be a congruence relation of L and let C be one of its congruence classes. Since in a distributive relatively complemented lattice each convex sublattice is congruence class of precisely one congruence relation [4] we have $R = R_C$. We show that Γ_C is the complement of R_C. $\Gamma_C \wedge R_C = \Omega$ was proved already. Let x,y be two arbitrary elements of L, and suppose $x \in C_{c_1}$, $y \in C_{c_2}$, with $c_1, c_2 \in C$. Then $x\ \Gamma_C c_1$, $c_1\ R_C c_2$ and $c_2\ \Gamma_C y$; hence, $x\ R_C \vee \Gamma_C y$, from which we conclude that $R_C \vee \Gamma_C = \mho$, i.e., Γ_C is the complement of $R_C = R$.

4. The lattice $(\mathcal{C}(L), \subset)$

Let L be a lattice. In this section we study some properties of the lattice $(\mathcal{C}(L), \subset)$ i.e., tha lattice of all convex sublattices of L, partially ordered by inclusion. The join operation in $(\mathcal{C}(L), \subset)$ is denoted by $\sqcup$ (definition 4.).

Lemma 4. Let L be a distributive lattice, and let $A, B, C \in \mathcal{C}(L)$. Then:

1. If $A \cap B \neq \emptyset$ and $A \cap C \neq \emptyset$, then $A \cap (B \sqcup C) = (A \cap B) \sqcup (A \cap C)$.
2. If $B \cap C \neq \emptyset$, then $A \sqcup (B \cap C) = (A \sqcup B) \cap (A \sqcup C)$.

Proof.

1. Clearly, $A \cap (B \sqcup C) \supset (A \cap B) \sqcup (A \cap C)$. In order to prove that $A \cap (B \sqcup C) \subset (A \cap B) \sqcup (A \cap C)$, assume that $a \in A$ and $a \in B \sqcup C$. This means that there exist $b_1, b_2 \in B$ and $c_1, c_2 \in C$ such that $b_1 \wedge c_1 \leq a \leq b_2 \vee c_2$. Let $s \in A \cap B$ and $t \in A \cap C$. Then: $a \leq a \vee s \vee t = (a \wedge b_2) \vee (a \wedge c_2) \vee s \vee t$. However, $(a \wedge b_2) \vee s \in A \cap B$ and $(a \wedge c_2) \vee t \in A \cap C$. Thus, $a \in \{(A \cap B) \vee (A \cap C)\}_1$. Similarly, $a \in \{(A \cap B) \wedge (A \cap C)\}_r$. This proves that $a \in (A \cap B) \sqcup (A \cap C)$.
2. Similar to part 1.

Theorem 6. Let L be a distributive relatively complemented lattice. Let C, D $\in \mathcal{C}(L)$ with $C \subset D$. There exists $C' \in \mathcal{C}(L)$ such that $C \cap C' = \emptyset$, $C \sqcup C' = D$.

Proof.

1. First we prove that for each $C \in \mathcal{C}(L)$ there exists C' such that $C \cap C' = \emptyset$ and $C \sqcup C' = L$. If C = L then $C' = \emptyset$. Otherwise, let $x \in L \setminus C$. Application of lemma 3 to the disjoint convex sublattices C and $\{x\}$ yields a prime ideal I, say a $\vee$-ideal, such that $C \cap I = \emptyset$. Since L is relatively complemented, I is maximal. We prove that $C \sqcup I = L$. $I \subset C \sqcup I \subset (C \sqcup I)_r$; hence, $I_r = L \subset (C \sqcup I)_r$. Thus, $(C \sqcup I)_r = L$, i.e. $C \sqcup I = (C \sqcup I)_l$. Since $I \subset (C \sqcup I)_l$ and since I is maximal, we have $(C \sqcup I)_l = I$ or $(C \sqcup I)_l = L$. $(C \sqcup I)_l = I$ contradicts $C \cap I = \emptyset$. We conclude therefore that $(C \sqcup I)_l = C \sqcup I = L$.
2. Let $C, D \in \mathcal{C}(L)$ with $C \subset D$. Since D is a convex sublattice, D is a relatively complemented (and distributive) lattice. We can therefore apply part 1, which yields a set C' such that:
   a. $C \cap C' = \emptyset$.
   b. The smallest convex sublattice of D that contains C and C' is D.
   c. C' is a $\vee$-ideal of D.
   From b. it follows that $C \sqcup C' = D$ (since each convex sublattice of L which is contained in D is a convex sublattice of D). Also, C' is a convex sublattice of L: It is clear that C' is a sublattice. Suppose that $c_1' \leq x \leq c_2'$, for some $c_1', c_2' \in C'$, $x \in L$. Since $c_1', c_2' \in C \subset D$, we have $x \in D$. Together with the fact that C' is a $\vee$-ideal of D and $x \leq c_2'$, this gives $x \in C'$; hence, C' is convex.

Theorem 6 asserts that if L is distributive relatively complemented lattice then each interval $[\emptyset, C]$ of $(\mathcal{C}(L), \subset)$ is complemented. Theorem 8 shows that an extra condition is necessary (and sufficient) in order that each interval $[C, D]$ of $(\mathcal{C}(L), \subset)$ be complemented (i.e., in order that $(\mathcal{C}(L), \subset)$ be relatively complemented).

For the proof of theorem 8 we need the following theorem of J. Hashimoto:

Theorem 7. The lattice of all $\vee$-ideals ($\wedge$-ideals) of a lattice L is distributive and relatively complemented if and only if L is distributive, relatively complemented and discrete.

Proof. See [6], theorem 4.3.

Theorem 8.

1. Let L be a distributive lattice $(\mathcal{G}(L), \subset)$ is relatively complemented if and only if L is relatively complemented and discrete.
2. Let L be a distributive lattice. Let $A,B,C \in \mathcal{G}(L)$ with $A \subset B \subset C$, $A \neq \emptyset$. Then: B has at most one complement in $[A,C]$.

Proof.

1.1. Suppose L is distributive, relatively complemented and discrete. Let A,B,C be elements of $\mathcal{G}(L)$ with $A \subset B \subset C$. We prove that there exists $B^* \in \mathcal{G}(L)$ such that $B \cap B^* = A$ and $B \sqcup B^* = C$. We may assume that $A \neq \emptyset$, since the case that $A = \emptyset$ was already treated in theorem 6. $A \subset B \subset C$ implies $A_1 \subset B_1 \subset C_1$ and $A_r \subset B_r \subset C_r$. Let $\mathfrak{I}(L)$ be the family of all $\vee$-ideals of L and $\mathfrak{J}(L)$ the family of all $\wedge$-ideals. By theorem 7, $(\mathfrak{I}(L), \subset)$ and $(\mathfrak{J}(L), \subset)$ are relatively complemented. Therefore, there exists $B_1^* \in \mathfrak{I}(L)$ such that $B_1 \cap B_1^* = A_1$ and such that C_1 is the smallest $\vee$-ideal that contains B_1 and B_1^*. Since L is distributive, this means that $b_1 \vee B_1^* = C_1$. Similarly, there exists $B_r^* \in \mathfrak{J}(L)$ such that $B_r \cap B_r^* = A_r$ and $B_r \wedge B_r^* = C_r$. We prove that $B_1^* \cap B_r^*$ is the relative complement of B in the interval $[A,C]$. Clearly, $B_1 \cap B_r \cap B_1^* \cap B_r^* = A_1 \cap A_r = A$. Also, $B \sqcup (B_1^* \cap B_r^*) = \{B \wedge (B_1^* \cap B_r^*)\}_r \cap \{B \vee (B_1^* \cap B_r^*)\}_1 = (B_r \wedge (B_1^* \cap B_r^*)_r) \cap (B_1 \vee (B_1^* \cap B_r^*)_1) = (B_r \wedge B_r^*) \cap (B_1 \vee B_1^*) = C_r \cap C_1 = C$.

1.2. Let L be distributive and suppose that $(\mathcal{G}(L), \subset)$ is relatively complemented. We show that then $\mathfrak{I}(L)$ is also relatively complemented. Theorem 7 then gives the desired result. Let $I_1 \subset I_2 \subset I_3$ be three elements of $\mathfrak{I}(L)$. There exists $C \in \mathcal{G}(L)$ such that $C \cap I_2 = I_1$, $C \sqcup_2 = I_3$. Since $I_1 \subset C$ we have $I_1 \subset C_r$; hence, $I_{12} = L \subset C_r$. This means that $C = C_1$; i.e., C is a $\vee$-ideal, from which we conclude that $\mathfrak{I}(L)$ is relatively complemented.

2. Let L be distributive, let $A \subset B \subset C \in \mathcal{G}(L)$ with $A \neq \emptyset$, and suppose that B has two relative complements B_1^* and B_2^* in $[A,C]$. As above, it follows that B_{1l}^* and B_{2l}^* are two relative complements (in $\mathcal{M}(L)$) of B_l in the interval $[A_l, C_l]$. Since $\mathcal{M}(L)$ is distributive, we have $B_{1l}^* = B_{2l}^*$. Similarly, $B_{1r}^* = B_{2r}^*$, whence $B_1^* = B_2^*$.

Remark: In the assertion that complementation in each interval $[A,C]$ of $\mathcal{G}(L)$ is unique (for L distributive), we may not omit the condition that $A \neq \emptyset$. This can be seen as follows: Suppose that complementation in the whole of $\mathcal{G}(L)$ is unique, for L distributive. If L is also relatively complemented and discrete, we would have: $(\mathcal{G}(L), \subset)$ is a lattice in which complements always exist and unique. Together with the atomicity of $(\mathcal{G}(L), \subset)$ this would give the result that $(\mathcal{G}(L), \subset)$ is distributive ([7], p. 57), which is clearly not the case.

<u>Corollary 1.</u> Let L be a distributive lattice, let $A,B \in \mathcal{G}(L)$ with $A \subset B$, $A \neq \emptyset$, and let $C(A,B)$ be the largest element of $\mathcal{G}(L)$ that has the intersection A with B (theorem 1). Then $B \sqcup C(A,B) = L$ for all A,B, if and only if L is relatively complemented and discrete.

<u>Proof</u>.

1. Let L be distributive relatively complemented and discrete. Let $A \subset B$, $A \neq \emptyset$ $(A,B \in \mathcal{G}(L))$, and let B^* be the complement of B in the interval $[A,L]$. Then $B \sqcap B^* = A$. By the definition of $C(A,B)$: $B^* \subset C(A,B)$; hence, $B \sqcup C(A,B) \supset B \sqcup B^* = L$.
   Thus, $B \sqcup C(A,B) = L$.
2. Let L be distributive and suppose that for each $A,B \in \mathcal{G}(L)$ with $A \subset B$, $A \neq \emptyset$, we have $B \sqcup C(A,B) = L$. In particular, if I and H are two $\vee$-ideals of L with $I \subset H$, we have $I \sqcup C(I,H) = L$. By theorem 1, $C(I,H) = (I_l \backslash (H \backslash I_r)_l)_r \cap (I_r \backslash (H \backslash I_l)_r)_l = (I \backslash (H \backslash L)_l)_r \cap (L \backslash (H \backslash I)_r)_l = L \cap (L \backslash (H \backslash I)_r)_l = (L \backslash (H \backslash I)_r)_l$. Thus, $C(I,H)$ is a $\vee$-ideal and we see that each interval $[I,L]$ of $\mathcal{M}(L)$ is complemented. Since (L and) $\mathcal{M}(L)$ is distributive, $\mathcal{M}(L)$ is relatively complemented. By theorem 7, L is then relatively complemented and discrete.

Corollary 2. Let L be a distributive relatively complemented and discrete lattice, let $A,B \in \mathcal{C}(L)$ with $A \subset B$, $A \neq \emptyset$. Let $C(A,B)$ be defined as in corollary 1. Then we have: $C(A,C(A,B)) = B$.

Proof. By corollary 1, we have

$$B \cap C(A,B) = A \quad \text{and} \quad B \sqcup C(A,B) = L,$$

$$C(A,C(A,B)) \cap C(A,B) = A \quad \text{and} \quad C(A,C(A,B)) \sqcup C(A,B) = L.$$

Uniqueness of complementation in $[A,L]$ yields $B = C(A,C(A,B))$.

5. The lattice $(\bar{\mathcal{C}}(L), \leq)$

In this section we study a partial ordering on $\bar{\mathcal{C}}(L)$ which is a variant of the ordering by inclusion. ($\bar{\mathcal{C}}(L)$ is used to denote the family of all non-empty convex sublattices of L).

Definition 7. Let L be a lattice and let $C,D \in \bar{\mathcal{C}}(L)$. We define the partial ordering $\leq$ as follows:

$$C \leq D \text{ if and only if } C \subset D_l \text{ and } D \subset C_r.$$

Lemma 5. $\leq$ is a partial ordering on $\bar{\mathcal{C}}(L)$.

Proof. We prove only anti-symmetry. Let $C,D \in \bar{\mathcal{C}}(L)$, with $C \leq D$ and $D \leq C$. Then $C \subset D_l$, $D \subset C_r$, $D \subset C_l$ and $C \subset D_r$. Hence, $C \subset D_l \cap D_r = D$ and $D \subset C_l \cap C_r = C$, which gives $C = D$.

Lemma 6. Let $C \in \bar{\mathcal{C}}(L)$. C is a $\vee$-ideal ($\wedge$-ideal) of L if and only if $C \leq L$ ($C \geq L$).

Proof.

1. From $C \leq L$ we see that $L \subset C_r$, whence $C = C_l \cap L = C_l$.
2. Let I be a $\vee$-ideal. Clearly, $I \subset L_l = L$. Also, $L \subset I_r$, since $L = I_{lr} = I_r$.

Lemma 7. Let $C,D \in \bar{\mathcal{C}}(L)$. Then $C \leq D$ if and only if $C \wedge D = C$ ($C \vee D = D$).

Proof. Follows directly from the definitions.

Theorem 9. Let L be a distributive lattice. Then $(\bar{\mathcal{C}}(L), \leq)$ is a distributive lattice.

Proof.

1. $C,D \in \bar{\mathcal{C}}(L)$ implies $C \wedge D \in \bar{\mathcal{C}}(L)$:
   a. Clearly, $(c_1 \wedge d_1) \wedge (c_2 \wedge d_2) \in C \wedge D$.
   b. $(c_1 \wedge d_1) \vee (c_2 \wedge d_2) = \{c_1 \vee (c_2 \wedge d_2)\} \wedge \{d_1 \vee (c_2 \wedge d_2)\} \in C \wedge D$.
   c. Suppose $c_1 \wedge d_1 \leq x \leq c_2 \wedge d_2$, for some $x \in L$. Then:
      $c_1 \leq x \vee c_1 \leq c_1 \vee (c_2 \wedge d_2)$; hence, $x \vee c_1 \in C$. Also, $x \vee d_1 \in D$,
      whence $x = x \vee (c_1 \wedge d_1) = (x \vee c_1) \wedge (x \vee d_1) \in C \wedge D$.
2. Similarly, $C \vee D \in \bar{\mathcal{C}}(L)$.
3. The commutative. associative and absorption laws follow directly.
4. Distributivity is proved by showing that, for $C,D,E \in \bar{\mathcal{C}}(L)$:
   $C \wedge (D \vee E) = (C \wedge D) \vee (C \wedge E)$.
   a. It is clear that $C \wedge (D \vee E) \subset (C \wedge D) \vee (C \wedge E)$.
   b. Let $(c_1 \wedge d) \vee (c_2 \wedge e) \in (C \wedge D) \vee (C \wedge E)$. Then:
      $(c_1 \wedge d) \vee (c_2 \wedge e) = \{(c_1 \wedge d) \vee c_2\} \wedge \{(c_1 \wedge d) \vee e\} =$
      $c_3 \wedge \{(c_1 \wedge d) \vee e\} = c_3 \wedge \{(c_1 \vee e) \wedge (d \vee e)\} = c_4 \wedge (d \vee e) \in C \wedge (D \vee E)$.

Corollary. Let L be a distributive lattice. Then $(\mathcal{Y}(L), \leq)$ is a $\vee$-ideal of $(\bar{\mathcal{C}}(L), \leq)$ and $(\mathcal{J}(L), \leq)$ is a $\wedge$-ideal of $(\bar{\mathcal{C}}(L), \leq)$.

Proof. Follows from lemma 6 and theorem 9.

In the remainder of this section we shall omit indication of the partial ordering $\leq$ on $\bar{\mathcal{C}}(L)$, i.e., when we write $\bar{\mathcal{C}}(L)$, we mean $(\bar{\mathcal{C}}(L), \leq)$.

Theorem 10. Let F_i $(i \geq 0)$ be the free distributive lattice with i generators, with an (extra) zero and unit element adjoined. Let $B_j (j \geq 1)$ be the Boolean algebra with 2^j elements. For L distributive, we define $\bar{\mathcal{C}}^0(L) = L$ and $\bar{\mathcal{C}}^i(L) = \bar{\mathcal{C}}(\bar{\mathcal{C}}^{i-1}(L))$, $(i \geq 0)$. Then we have: $\bar{\mathcal{C}}^i(B_j)$ is isomorphic with the direct union of j factors F_i (cf. [1], chapter IX, section 10).

Proof. We use induction on i.

1. $\bar{\mathcal{C}}^0(B_j)$ is clearly isomorphic with the direct union of j factors F_0,

since $F_0 \cong B_1$ [1])

2. Suppose $\bar{\mathcal{C}}^i(B_j) \cong F_i^j$ (The direct union of two lattices L_1, L_2 is denoted by $L_1 \times L_2$; the direct union of j factors L is denoted by L^j). In order to prove that $\bar{\mathcal{C}}^{i+1}(B_j) \cong F_i^{j+1}$, we have to prove that $\bar{\mathcal{C}}(F_i^j) \cong F_{i+1}^j$. However, it is easy to verify that for two finite distributive lattices L_1, L_2 we have $\bar{\mathcal{C}}(L_1 \times L_2) \cong \bar{\mathcal{C}}(L_1) \times \bar{\mathcal{C}}(L_2)$. Therefore, there remains the proof of $\bar{\mathcal{C}}(F_i) \cong F_{i+1}$. Let $C = \{f_i \in F_i \mid a \leq f_i \leq b\}$ be an element of $\bar{\mathcal{C}}(F_i)$, where a and b are finite joins of meets of the generators, say $x_1, x_2, \ldots, x_i$, of F_i. (Verification of the following argument in the case that a or b is the zero or unit element of F_i is straight forward and is therefore omitted). We define the isomorphism $\psi: \bar{\mathcal{C}}(F_i) \to F_{i+1}$ as follows: We introduce $y (\neq x_1, x_2, \ldots, x_i)$ as the i+1-th generator of F_{i+1}. Consider the element $(b \wedge y) \vee a$ of F_{i+1}. It may be possible to "reduce" this element: E.g., let $b = x_1 \vee x_2$, and $a = x_1$. Then $(b \wedge y) \vee a = ((x_1 \vee x_2) \wedge y) \vee x_1$ can be reduced to $(x_2 \wedge y) \vee x_1$. Clearly, however, each element $(b \wedge y) \vee a$ has an "irreducible" form. From now on we assume that all elements of F_{i+1} are in reduced form. We then define $\psi(C)$ as $(b \wedge y) \vee a$. We prove that ψ is an isomorphism:

Let $C_1 = \{f_i \in F_i \mid a \leq f_i \leq b\}$ and $C_2 = \{f_i \in F_i \mid c \leq f_i \leq d\}$. Then:

$C_1 \wedge C_2 = \{f_i \in F_i \mid a \wedge c \leq f_i \leq b \wedge d\}$, and

$C_1 \vee C_2 = \{f_i \in F_i \mid a \vee c \leq f_i \leq b \vee d\}$.

$\psi(C_1) \wedge \psi(C_2) = \{(b \wedge y) \vee a\} \wedge \{(d \wedge y) \vee c\} =$

$(b \wedge d \wedge y) \vee (a \wedge d \wedge y) \vee (b \wedge c \wedge y) \vee (a \wedge c) = (b \wedge d \wedge y) \vee (a \wedge c) = \psi(C_1 \wedge C_2)$

$\psi(C_1) \vee \psi(C_2) = \{(b \wedge y) \vee a\} \vee \{(d \wedge y) \vee c\} = \{(b \vee d) \wedge y\} \vee (a \vee c) = \psi(C_1 \vee C_2)$

Suppose $\psi(C_1) = \psi(C_2)$. This means that $(b \wedge y) \vee a = (d \wedge y) \vee c$.

From the irreducibility of $(b \wedge y) \vee a$ and $(d \wedge y) \vee c$, it follows that $a = c$ and $b = d$. Hence, ψ is 1-1.

Also, ψ is onto: Each element of F_{i+1} can be written as $(a \wedge y) \vee b$, with $a, b \in F_i$. However, $(a \wedge y) \vee b$ is the image of the convex sublattice $\{f_i \in F_i \mid b \leq f_i \leq a \vee b\}$ of F_i. This follows from $\{(a \vee b) \wedge y\} \vee b = (a \wedge y) \vee b$.

This completes the proof of theorem 10.

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1) $\cong$ is used to denote isomorphism.

Remark: Let K_n be the chain of n elements. We state without proof the following formulae:

Let $\gamma_n^{(i)}$ be the number of elements of $\bar{\mathcal{C}}^{(i)}(K_n)$, i=1,2,3.

Then $\gamma_n^{(1)} = \sum_{i=1}^{n} (n-i+1) = \frac{1}{2}n(n+1)$,

$$\gamma_n^{(2)} = \sum_{i=1}^{n} (n-i+1)i^2 = \frac{1}{12}\, n(n+1)^2(n+2)$$

$$\gamma_n^{(3)} = \frac{1}{120} \sum_{i=1}^{n} (n-i+1)(8i^6 + 24i^5 + 35i^4 + 30i^3 + 17i^2 + 6i).$$

6. A ternary function in distributive relatively complemented lattices

Theorem 11. Let L be a distributive relatively complemented lattice. Let $f: L^3 \to L$ be defined as follows: f(a,b,c) is the relative complement of a in the interval $[a \wedge b,\ a \vee c]$. Then we have:
A subset C of L is an element of $\mathcal{C}(L)$ if and only if $f(L,C,C) \subset C$.

Proof.

1. Suppose that $f(L,C,C) \subset C$. Clearly, $f(c_1,c_2,c_1) = c_1 \wedge c_2 \in C$, and $f(c_1,c_1,c_2) = c_1 \vee c_2 \in C$; hence, C is a sublattice. Also, if $c_1 \leq x \leq c_2$, then $x = f(x, c_2, c_1) \in C$.
2. Suppose $C \in \mathcal{C}(L)$. Let $f(x, c_1, c_2)$ be an element of f(L,C,C).
   We prove that $f(x, c_1, c_2)$(abbreviated to x^*) is an element of C.
   We have $x \wedge x^* = x \wedge c_1$ and $x \vee x^* = x \vee c_2$.
   Hence,
   $x^* = x^* \vee (x \wedge c_1) = (x^* \vee x) \wedge (x^* \vee c_1) = (x \vee c_2) \wedge (x^* \vee c_1) = \{x \wedge (x^* \vee c_1)\} \vee \{c_2 \wedge (x^* \vee c_1)\} = \{(x \wedge x^*) \vee (x \wedge c_1)\} \vee c_3 = (x \wedge c_1) \vee c_3 \in C.$

Theorem 12. A set L is a distributive relatively complemented lattice if and only if there exists a function $f: L^3 \to L$ with the following properties: For all $a,b,c,d,e \in L$:

P1. f(a,a,a) = a.

P2.1. f(a,b,a) = f(b.a.b)

P2.2. f(a,a,b) = f(b,b,a).

P3. $f(a,f(a,b,c), f(a,d,e)) = f(a,b,e)$,

P4.1. $f(a,f(b,b,c),a) = f(f(a,b,a), f(a,b,a), f(a,c,a))$.

P4.2. $f(a,a,f(b,b,c)) = f(f(a,a,b), f(a,a,b), f(a,a,c))$.

Proof.

1. The condition is sufficient.

We define $a \wedge b = f(a,b,a)$ and $a \vee b = f(a,a,b)$.

1.1. The commutativity of $\wedge$ and $\vee$ follows from P2.

1.2. $a \wedge (a \vee b) = f(a, f(a,a,b),a) = f(a, f(a,a,b), f(a,a,a)) =$
$= f(a,a,a) = a$, by P1, P3 and P1.

Similarly, $a \vee (a \wedge b) = a$.

1.3. In order to prove that $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, we have to show that:

$f(a, f(b,c,b), a) = f(f(a,b,a), c, f(a,b,a))$.

Let $A = f(a, f(b,c,b), a)$ and $B = f(f(a,b,a), c, f(a,b,a))$.

First we prove that $a \wedge A = a \wedge B$ and $a \vee A = a \vee B$:

$f(a,A,a) = f(a, f(a, f(b,c,b), a), a) = f(a, f(b,c,b), a)$ by P3 and P1.

$$\begin{aligned} f(a,B,a) &= f(a, f(a,b,a), c, f(a,b,a)), a) \\ &= f(f(a, f(a,b,a), a), f(a,c,a), f(a, f(a,b,a), a)) \\ &= f(f(a,b,a), f(a,c,a), f(a,b,a)) \\ &= f(a, f(b,c,b), a) \text{ by P4.1, P1 and P3, and P4.1.} \end{aligned}$$

$f(a,a,A) = f(a,a, f(a, f(b,c,b), a)) = a$, by P1 and P3.

$$\begin{aligned} f(a,a,B) &= f(a, a, f(f(a,b,a), c, f(a,b,a))) \\ &= f(f(a, a, f(a,b,a)), f(a,a,c), f(a,a, f(a,b,a))) \\ &= f(a, f(a,a,c), a) = a, \text{ by P4.2, P1 and P3.} \end{aligned}$$

The rest of the proof that $A = B$ is standard:

$$\begin{aligned} A &= A \vee (A \wedge a) = A \vee (B \wedge a) = (A \vee B) \wedge (A \vee a) \\ &= (A \vee B) \wedge (B \vee a) = B \vee (a \wedge A) = B \vee (a \wedge B) = B, \end{aligned}$$

by application of P2 and P4.2.

For the proof of $(a \vee b) \vee c = a \vee (b \vee c)$ we need the dual equalities of P4.1 and P4.2, i.e:

(1) $f(a, f(b,b,c), a) = f(f(a,b,a), f(a,b,a), f(a,c,a))$ and

(2) $f(a, a, f(b,b,c)) = f(f(a,a,b), f(a,a,b), f(a,a,c))$.

(1) is established as usual:

$$(a\wedge b)\vee(a\wedge c) = \{(a\wedge b)\vee a\}\wedge\{(a\wedge b)\vee c\}$$
$$= a\wedge\{(a\wedge b)\vee c\} = a\wedge\{(a\vee c)\wedge(b\vee c)\}$$
$$= \{a\wedge(a\vee c)\}\wedge(b\vee c) = a\wedge(b\vee c),$$

by the associativity of $\wedge$ and P4.2.

To prove (2), we consider:

$$\{a\vee(b\vee c)\}\wedge\{(a\vee b)\vee(a\vee c)\} =$$
$$[a\wedge\{(a\vee b)\vee(a\vee c)\}]\vee[(b\vee c)\wedge\{(a\vee b)\vee(a\vee c)\}] =$$
$$[\{a\wedge(a\vee b)\}\vee\{a\wedge(a\vee c)\}]\vee[(b\wedge\{(a\vee b)\vee(a\vee c)\})\vee$$
$$(c\wedge\{(a\vee b)\vee(a\vee c)\})] = a\vee[(\{b\wedge(a\vee b)\}\vee\{b\wedge(a\vee c)\})$$
$$\vee(\{c\wedge(a\vee b)\}\vee\{c\wedge(a\vee c)\})] = a\vee[(b\vee\{b\wedge(a\vee c)\})\vee$$
$$(\{c\wedge(a\vee b)\}\vee c)] = a\vee(b\vee c)$$

and:

$$\{a\vee(b\vee c)\}\wedge\{(a\vee b)\vee(a\vee c)\} =$$
$$[\{a\vee(b\vee c)\}\wedge(a\vee b)]\vee[\{a\vee(b\vee c)\}\wedge(a\vee c)] =$$
$$[a\vee\{(b\vee c)\wedge b\}]\vee[a\vee\{(b\vee c)\wedge c\}] =$$
$$(a\vee b)\vee(a\vee c).$$

Hence, $a\vee(b\vee c) = (a\vee b)\vee(a\vee c)$.
Finally, the proof of the associativity of $\vee$ is now dual to the proof of the associativity of $\wedge$.

1.4. The distributivity of L follows from P4.2 and (1).

1.5. Let $a \leq c \leq b$. Then:
$c\wedge f(c,a,b) = f(c, f(c,a,b),c) = f(c,a,c) = c\wedge a = a$ and
$c\vee f(c,a,b) = f(c, c, f(c,a,b)) = f(c,c,b) = c\vee b = b$.
Thus, $f(c,a,b)$ is the relative complement of c in the interval $[a,b]$.

2. The condition is necessary.
Let L be a distributive relatively complemented lattice. Let $f(a,b,c)$ be the relative complement of a in the interval $[a\wedge b,\ a\vee c]$.
Then $f(a,b,c)$ has the properties P1 to P4.
We prove only P2.1 and P3.
Clearly, the relative complement of a in the interval $[a\wedge b,\ a]$ is $a\wedge b$. Thus, $f(a,b,a) = a\wedge b = b\wedge a = f(b,a,b)$.

Furthermore, by the definition of f,

$a \wedge f(a,\ f(a,b,c),\ f(a,d,e)) = a \wedge f(a,b,c) = a \wedge b$ and

$a \wedge f(a,b,e) = a \wedge b$.

$a \vee f(a,\ f(a,b,c),\ f(a,d,e)) = a \quad f(a,d,e) = a \vee e$ and

$a \vee f(a,b,e) = a \vee e$.

Hence, $f(a,\ f(a,b,c),\ f(a,d,e)) = f(a,b,e)$.

This completes the proof of theorem 12.

Finally we mention some properties of the function f that can be verified directly from its definition:

P5. $f(b,a,a) = a$.

P6. $f(a,b,f(a,b,c)) = f(a,\ f(a,b,c),\ c) = f(\ f(\ f(a,b,c),b,c),b,c) = f(a,b,c)$.

P7. $f(f(a,b,c),b,c) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$.

P8. $f(a,\ f(b,c,d),\ f(b,e,g)) = f(b,\ f(a,c,e),\ f(a,d,g))$.

Remarks:

1. The function f has been used to define Boolean algebra's and distributive relatively complemented lattices with zero in [3].
2. From P7 we see that the function f is related to the well-known ternary function $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$, which has been used for the axiomatics of distributive lattices by several authors (these investigations started with [5]; for recent results see [9]).
3. From P3, P5 and P8 we see that f is one of the "selection functions" as studied in [2]. In particular, if L is the Boolean algebra with two elements, then f coincides with the "conditional Boolean expression" <u>if</u> a <u>then</u> b <u>else</u> c, as used in the programming language ALGOL 60 [8].

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